

Symmetries, Integrability and Exact Solutions for Nonlinear Systems

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Abstract

The paper intends to offer a general overview on what the concept of integrability means for a nonlinear dynamical system and how the symmetry method can be applied for approaching it. After a general part where key problems as direct and indirect symmetry method or optimal system of solutions are tackled out, in the second part of the lecture two concrete models of nonlinear dynamical systems are effectively studied in order to illustrate how the procedure is working out. The two models are the 2D Ricci flow model coming from the general relativity and the 2D convective-diffusion equation. Part of the results, especially concerning the optimal systems of solutions, are new ones.

Keywords: Lie symmetries, invariants, similarity reduction.

1 Integrability and symmetries. Key aspects.

1.1 The concept of integrability for dynamical systems

Dynamical systems described by nonlinear partial differential equations are frequently used to model a wide variety of phenomena in physics, chemistry, biology and other fields [1]. The modelling process includes to find solutions of those partial differential equations. If these solutions exist, the differential system is said to be integrable. Sometime it is difficult to find a complete set of solutions and it is quite enough if one can decide on the integrability of the system. There are many methods which can be used to fulfill this aim: the Hirota's bilinear method, the Backlund transformation method, the inverse scattering method, the Lax pair operator, the Painleve analysis and others [2]. Each method has its own significant properties. For example, while the Lax and the Painleve methods are moreover testing the integrability, the Hirota's bilinear method is very efficient for the effective determination of the multiple soliton solutions for a wide class of nonlinear evolution equations [3]. As a conclusion, to decide that a nonlinear differential equation is integrable, one of the following situation should appear:

(i) the existence of a number of functionally independent first integrals/invariants equal to the order of the system in general and half that for a Lagrangian system as a consequence of Liouville's Theorem;

(ii) the existence of a sufficient number of Lie symmetries to reduce the partial differential equation to an ordinary differential equation;

(iii) the possession of the Painlevé property [4].

In this lecture the first two criteria will be investigated.

1.2 The symmetry method for solving dynamical systems

Many natural phenomena are described by a system of nonlinear partial differential equations (pdes) which is often difficult to be solved analytically, as there is no a general theory for completely solving of the nonlinear pdes. One of the most useful techniques for finding exact solutions of the dynamical

systems described by nonlinear pdes is *the symmetry method*. On the one hand, one can consider symmetry reduction of differential equations and thus obtain classes of exact solutions. On the other hand, by definition, a symmetry transforms solutions into solutions, and thus symmetries can be used to generate new solutions from known ones.

Initially the symmetry method for solving partial differential equations was developed for what is currently known as the *Lie (classical) symmetry method (CSM)*. We shall present now a short introduction to this approach [5].

Let us consider a n -th order partial differential system:

$$\Delta_\nu(x, u^{(n)}[x]) = 0 \quad (1)$$

where $x \equiv \{x^i, i = \overline{1, p}\} \subset R^p$ represent the independent variables, while $u \equiv \{u^\alpha, \alpha = \overline{1, q}\} \subset R^q$ the dependent ones. The notation $u^{(n)}$ designates the set of variables which includes u and the partial derivatives of u up to n -th order.

The general infinitesimal symmetry operator has the form:

$$U = \sum_{i=1}^p \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (2)$$

The n -th extension of (2) is given by:

$$U^{(n)} = U + \sum_{\alpha=1}^q \sum_J \phi_\alpha^J(x, u^{(n)}) \frac{\partial}{\partial u_J^\alpha} \quad (3)$$

where

$$u_J^\alpha = \frac{\partial^m u^\alpha}{\partial x^{j_1} \partial x^{j_2} \dots \partial x^{j_m}} \quad (4)$$

Also, in (4) the second summation refers to all the multi-indices $J = (j_1, \dots, j_m)$, with $1 \leq j_m \leq p, 1 \leq m \leq n$. The coefficient functions ϕ_α^J are given by the following formula:

$$\phi_\alpha^J(x^i, u^{(n)}) = \mathcal{D}_J[\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha] + \sum_{i=1}^p \xi^i u_{J,i}^\alpha, \quad \alpha = \overline{1, q} \quad (5)$$

in which

$$u_i^\alpha = \frac{\partial u^\alpha}{\partial x^i}, \quad i = \overline{1, p} \quad (6)$$

$$u_{J,i}^\alpha = \frac{\partial u_J^\alpha}{\partial x^i} = \frac{\partial^{m+1} u^\alpha}{\partial x^i \partial x^{j_1} \partial x^{j_2} \dots \partial x^{j_m}} \quad (7)$$

$$\mathcal{D}_J = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \dots \mathcal{D}_{j_m} = \frac{d^m}{dx^{j_1} dx^{j_2} \dots dx^{j_m}} \quad (8)$$

The Lie symmetries represent the set of all the infinitesimal transformations which keep invariant the differential system. The invariance condition is:

$$U^{(n)}[\Delta] \big|_{\Delta=0} = 0 \quad (9)$$

The characteristic equations associated to general symmetry generator (2) have the form:

$$\frac{dx^1}{\xi^1} = \dots = \frac{dx^p}{\xi^p} = \frac{du^1}{\phi_1} = \dots = \frac{du^q}{\phi_q} \quad (10)$$

By integrating the characteristic system of ordinary differential equations (10), the invariants I_r , $r = 1, (p + q - 1)$ of the analyzed system can be found. They are identified with the

constants of integration. Following this way, the set of similarity variables is found in terms of which the original evolutionary equation with p independent variables and q dependent ones can be reduced to a set of differential equations with $(p + q - 1)$ variables. These are the similarity reduced equation which generate the similarity solution of the analyzed model.

There have been *several generalizations* of the Lie symmetry method which include:

- 1) the *non-classical symmetry method* (NSM) (also referred to as *the conditional method*) of Bluman and Cole [6],
- 2) the *direct method* of Clarkson and Kruskal [11],
- 3) the *differential constraint approach* of Olver and Rosenau [12]
- 4) the *generalized conditional symmetry* method due to Fokas, Liu and Zhdanov [13].

The basic idea of *the nonclassical method* is that (12) should be augmented with the invariance surface condition:

$$Q^\alpha(x, u^{(1)}) \equiv \phi_\alpha(x, u) - \sum_{i=1}^p \xi^i(x, u) \frac{\partial u^\alpha}{\partial x^i} = 0, \quad \alpha = \overline{1, q} \quad (11)$$

The q -tuple $Q = (Q^1, Q^2, \dots, Q^q)$ is known as the characteristic of the symmetry operator (2). The invariance condition (9) must be applied taking into account that the constraints (11) do exist. The number of determining equations for the infinitesimals $\xi^i(x, u)$, $\phi_\alpha(x, u)$, appearing in the nonclassical method is smaller than for the classical method. The main difficulty of this approach is that the determining equations are no longer linear. On the other hand, the NSM may produce more solutions than the CSM, since any classical symmetry is a nonclassical one, but not conversely.

The *direct method* represents a direct, algorithmic, and nongroup theoretic method for finding symmetry reductions. The relationship between this direct method and the nonclassical method has been discussed in many papers (e.g., [7], [8]). In particular, Levi and Winternitz [9] established, using a group-theoretic explanation, that all new solutions obtained by the direct method can also be obtained by the nonclassical method. In fact, it has been shown in [10] that the similarity solutions corresponding to the nonclassical groups should in general constitute a larger family than that obtained by the direct method.

The *differential constraint approach* proposed a generalization of the nonclassical method. Its promoters shown that many known reduction methods, including the classical and nonclassical methods, partial invariance, and separation of variables can be placed into a general framework. In their formulation, the original system of partial differential equations can be enlarged by appending additional differential constraints (side conditions), such that the resulting overdetermined system of partial differential equations satisfying compatibility conditions.

As well, in further efforts to find new symmetries of PDEs which would lead to additional new invariant solutions, much work has been done in the area of higher-order symmetries. In particular, for an evolution equation in two independent variables and one dependent variable has been introduced in [13] the method of *generalized conditional symmetries* (GCS) or conditional Lie-Bäcklund symmetries.

1.3 Optimal system of solutions

In general, when a differential equation admits a Lie group \mathcal{G}_r and its Lie algebra \mathcal{L}_r is of dimension $r > 1$, one desires to minimize the search for invariant solutions by finding the nonequivalent branches of solutions. This leads to the concept of *optimal system*.

It is well known that for one-dimensional subalgebras, the problem of finding an optimal system of subalgebras is essentially the same as the problem of classifying the orbits of the adjoint transformations.

In Ovsiannikov [14], the *global matrix of the adjoint transformations* is used in constructing the one-dimensional optimal system.

In Olver [5], a slightly different technique is employed: it consists in constructing a table, named the *adjoint table*, showing the separate adjoint actions of each element in \mathcal{L}_r as it acts on all the other elements .

The procedure reported in Ruggieri and Valenti [15], is a mixed of the above procedures and consists in constructing the *global matrix of the adjoint transformations* by means of the *adjoint table*.

One of the advantages of the symmetry analysis is the possibility to find solutions of the original pdes by solving odes. These odes, called *reduced equations*, are obtained by introducing suitable new variables, determined as invariant functions with respect to the infinitesimal generators.

On the basis of the infinitesimal generators of the optimal systems of the Lie algebras of analyzed model, we can construct the reduced odes of the given model and find exact solutions.

1.4 The inverse Lie symmetry problem

Usually, the *direct symmetry problem* of evolutionary equations is considered for finding their exact solutions. It also known as the classical symmetry method. Firstly, it consists in determining the Lie symmetry group corresponding to a given evolutionary equation. Then, using the characteristic equations could be obtained the Lie invariants associated to each symmetry operator. Further these invariants, following the reduced similarity procedure, determine the reduced equation which could be solve and generates the similarity solution of the analyzed model.

Also, the *inverse symmetry problem* [16] could be made. We ask the question: what is the largest class of evolutionary equations which are equivalent from the point of view of their symmetries?. So, this problem could be solved by imposing a concrete symmetry group to a general analyzed model. With this condition, the general symmetry determining equations could be solved and allow to determine all concrete models which admit the same Lie symmetry group.

Let us consider a 2D dynamical system described by a second order partial derivative equation of the general form:

$$u_t = A(x, y, t, u)u_{xy} + B(x, y, t, u)u_xu_y + C(x, y, t, u)u_{2x} + D(x, y, t, u)u_{2y} + E(x, y, t, u)u_y + F(x, y, t, u)u_x + G(x, y, t, u) \quad (12)$$

with $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ arbitrary functions of their arguments.

The general expression of the Lie symmetry operator which leaves (12) invariant is:

$$U(x, y, t, u) = \varphi(x, y, t, u)\frac{\partial}{\partial t} + \xi(x, y, t, u)\frac{\partial}{\partial x} + \eta(x, y, t, u)\frac{\partial}{\partial y} + \phi(x, y, t, u)\frac{\partial}{\partial u} \quad (13)$$

Through loss the generality we can choose in the previous expression $\varphi \equiv 1$. Then, the generator (13) becomes:

$$U(x, y, t, u) = \frac{\partial}{\partial t} + \xi(x, y, t, u)\frac{\partial}{\partial x} + \eta(x, y, t, u)\frac{\partial}{\partial y} + \phi(x, y, t, u)\frac{\partial}{\partial u} \quad (14)$$

Following the symmetry theory [5], the second extension $U^{(2)}$ of (13) has to be considered and the invariance condition of the equation (12) is given by the relation:

$$0 = U^{(2)}[u_t - A(x, y, t, u)u_{xy} - B(x, y, t, u)u_xu_y - C(x, y, t, u)u_{2x} - D(x, y, t, u)u_{2y} - E(x, y, t, u)u_y - F(x, y, t, u)u_x - G(x, y, t, u)] \quad (15)$$

The previous relation has the equivalent expression:

$$\begin{aligned}
0 = & -A_t u_{xy} - B_t u_x u_y - C_t u_{2x} - D_t u_{2y} - E_t u_y - F_t u_x - G_t - A_x \xi u_{xy} - B_x \xi u_x u_y - \\
& -C_x \xi u_{2x} - D_x \xi u_{2y} - E_x \xi u_y - F_x \xi u_x - G_x \xi - A_y \eta u_{xy} - B_y \eta u_x u_y - C_y \eta u_{2x} - D_y \eta u_{2y} - \\
& -E_y \eta u_y - F_y \eta u_x - G_y \eta - A_u \phi u_{xy} - B_u \phi u_x u_y - C_u \phi u_{2x} - D_u \phi u_{2y} - E_u \phi u_y - F_u \phi u_x - \\
& -G_u \phi + \phi^t - A\phi^{xy} - C\phi^{2x} - D\phi^{2y} - B\phi^x u_y - F\phi^x - B\phi^y u_x - E\phi^y
\end{aligned} \tag{16}$$

The functions $\phi^t, \phi^x, \phi^y, \phi^{2x}, \phi^{2y}, \phi^{xy}$ will be determined using the general formulas:

$$\begin{aligned}
\phi^t &= \mathcal{D}_t[\phi - u_t - \xi u_x - \eta u_y] + u_{2t} + \xi u_{xt} + \eta u_{yt} \\
\phi^x &= \mathcal{D}_x[\phi - u_t - \xi u_x - \eta u_y] + u_{tx} + \xi u_{2x} + \eta u_{xy} \\
\phi^y &= \mathcal{D}_y[\phi - u_t - \xi u_x - \eta u_y] + u_{ty} + \xi u_{xy} + \eta u_{2y} \\
\phi^{xy} &= \mathcal{D}_{xy}[\phi - u_t - \xi u_x - \eta u_y] + u_{txy} + \xi u_{xxy} + \eta u_{xyy} \\
\phi^{2x} &= \mathcal{D}_{2x}[\phi - u_t - \xi u_x - \eta u_y] + u_{txx} + \xi u_{xxx} + \eta u_{xxy} \\
\phi^{2y} &= \mathcal{D}_{2y}[\phi - u_t - \xi u_x - \eta u_y] + u_{tyy} + \xi u_{xyy} + \eta u_{yyy}
\end{aligned} \tag{17}$$

By extending the relations (17), substituting them into the condition (16) and then equating with zero the coefficient functions of various monomials in derivatives of u , the following partial differential system with 11 equations is obtained:

$$\begin{aligned}
0 &= \xi_u \\
0 &= \eta_u \\
0 &= B\eta_x - D\phi_{2u} \\
0 &= B\xi_y - C\phi_{2u} \\
0 &= A\eta_y - \eta A_y - A_u \phi + A\xi_x - \xi A_x + 2D\xi_y + 2C\eta_x - A_t \\
0 &= A\eta_x + 2D\eta_y - \eta D_y - \xi D_x - D_u \phi - D_t \\
0 &= -A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_t - B_x \xi - B_u \phi - B_y \eta \\
0 &= -\eta_t + F\eta_x - B\phi_x + E\eta_y - E_t - E_x \xi - E_y \eta - E_u \phi \\
&\quad + A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu} \\
0 &= -\xi_t - B\phi_y + F\xi_x + E\xi_y - F_t - F_x \xi - F_y \eta - F_u \phi \\
&\quad + A\xi_{xy} - A\phi_{yu} + C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu} \\
0 &= \phi_t + G\phi_u - F\phi_x - E\phi_y - G_t - G_x \xi - G_y \eta - G_u \phi \\
&\quad - A\phi_{xy} - C\phi_{2x} - D\phi_{2y}
\end{aligned} \tag{18}$$

The number of equations and of unknown functions which appear in the system (18) is relatively high. Two approaches are now possible: (i) to find the symmetries of a given evolutionary equation, which means to choose concrete forms for $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ and to use the system (18) in order to find the coefficient functions $\xi(x, y, t)$, $\eta(x, y, t)$ and $\phi(x, y, t, u)$ of the Lie operator; (ii) to solve the system (18) taking as unknown variables $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ and imposing a concrete form of the symmetry group. The first approach represents the *direct symmetry problem* and it is the usual one followed in the study of the Lie symmetries of a given dynamical system. The second approach, (ii), represents the *inverse symmetry problem* and it is more special, allowing us to determine all the equations which are equivalent from the point of view of the symmetry group they do admit.

2 Applications

In the next considerations, we will solve the direct and inverse Lie symmetry problems for two $2D$ nonlinear models: the Ricci flow model and the convective-diffusion equation.

2.1 The Lie symmetry problems for 2D Ricci flow model

One of the most fruitful models used in study of the black holes and in the attempt of obtaining a quantum theory of gravity is connected with the *Ricci flow equations* [17].

We will investigate a $2D$ model for the Ricci flow equation, a nonlinear parabolic equation obtained when the components of the metric tensor $g_{\alpha\beta}$ are deformed following the equation:

$$\frac{\partial}{\partial t} g_{\alpha\beta} = -R_{\alpha\beta} \quad (19)$$

where $R_{\alpha\beta}$ is the Ricci tensor for the n -dimensional Riemann space. The metric tensor of the space $g_{\alpha\beta}$ will be connected with the Riemann metric in the conformal gauge:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{2} \exp\{\Phi(X, Y, t)\} (dX^2 + dY^2) \quad (20)$$

The "potential" $\Phi(X, Y, t)$ satisfies the equation:

$$\frac{\partial}{\partial t} e^\Phi = \Delta \Phi \quad (21)$$

It has been noticed [18] that the equation (21) is pretty similar with the Toda equation describing the integrable interaction of a collection of two dimensional fields $\{\Phi_i, i = 1, 2\}$ coupled by a Cartan matrix (K_{ij}):

$$\sum_j K_{ij} e^{\Phi_j(X, Y)} = \Delta \Phi_i(X, Y) \quad (22)$$

Introducing the field $u(x, y, t)$ given by

$$u(x, y, t) = e^\Phi \quad (23)$$

the equation (21) takes the form:

$$u_t = (\ln u)_{xy} \quad (24)$$

An equivalent form for the previous equation, which will be used in the next considerations of the paper, is:

$$u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2} \quad (25)$$

The previous equation could be derived from the general one (12) by choosing the following particular coefficient functions:

$$\begin{aligned} A(x, y, t, u) &= \frac{1}{u}, B(x, y, t, u) = -\frac{1}{u^2}, \\ C(x, y, t, u) &= D(x, y, t, u) = E(x, y, t, u) = F(x, y, t, u) = G(x, y, t, u) \equiv 0 \end{aligned} \quad (26)$$

2.1.1 Lie symmetries for 2D Ricci flow model

To finding the *Lie symmetry operators* for the Ricci flow model (25) we have to solved the general determining system (18) in the conditions (26). The solution is represented by the coefficient functions $\xi(x, y, t)$, $\eta(x, y, t)$, $\phi(x, y, t, u)$ which determine the Lie symmetry operator (13). It has the form:

$$U = \frac{\partial}{\partial t} + \xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y} - u[\xi_x(x) + \eta_y(y)] \frac{\partial}{\partial u} \quad (27)$$

As U contains coefficients in the form of two arbitrary functions $\{\xi, \eta\}$, we deal with an infinite number of symmetry operators. The action of U can be split in various "sectors", depending on the concrete form we might choose for these functions.

Let us consider the *linear sector* of the Lie symmetries in which the forms of the coefficient functions of the symmetry generator (13) are:

$$\varphi = 1, \quad \xi = mx + c_1, \quad \eta = vy + c_2, \quad \phi = -(m + v)u \quad (28)$$

with m, v, k, c_1, c_2, c_3 arbitrary constants.

The general Lie symmetry operator (13) becomes:

$$U(x, y, t, u) = \frac{\partial}{\partial t} + (mx + c_1) \frac{\partial}{\partial x} + (vy + c_2) \frac{\partial}{\partial y} - (m + v)u \frac{\partial}{\partial u} \quad (29)$$

Consequently, the nonlinear Ricci flow equation (25) admits the 4-dimensional Lie algebra spanned by the independent operators shown below:

$$V_1 = x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u}, \quad V_4 = \frac{\partial}{\partial y} \quad (30)$$

The forms of the operators V_i , $i = \overline{1, 4}$ suggest their significations: V_2, V_4 generate the symmetry of space translations, V_1, V_3 are associated with the scaling transformations.

When the Lie algebra of these operators is computed, the only non-vanishing relations are:

$$[V_2, V_1] = V_2, \quad [V_4, V_3] = V_4 \quad (31)$$

2.1.2 Optimal system of subalgebras for 2D Ricci flow model

It is well known that reduction of the independent variables by one is possible using any linear combination of the generators of symmetry (30) $V_i, i = \overline{1, 4}$. We will construct a set of minimal combinations known as optimal system [5]. To construct the optimal system we need the commutators of the admitted symmetries given in the Table 1.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	$-V_2$	0	0
V_2	V_2	0	0	0
V_3	0	0	0	$-V_4$
V_4	0	0	V_4	0

Table1: Lie brackets of the admitted symmetry algebra

An optimal system of a Lie algebra is a set of l -dimensional subalgebras such that every l -dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation. The adjoint representation of a Lie algebra $\{V_i, i = 1, \dots, 4\}$ is constructed using the formula [5]:

$$Ad(\exp(\varepsilon V_i))V_j = \sum_n \frac{\varepsilon^n}{n!} (ad V_i)^n V_j = V_j - \varepsilon [V_i, V_j] + \frac{\varepsilon^2}{2!} [V_i, [V_i, V_j]] - \dots \quad (32)$$

Let us consider the linear combination of the symmetry generators:

$$V = a_1 V_1 + a_2 V_2 + a_3 V_3 + a_4 V_4 \quad (33)$$

Our task is to simplify as many of the coefficients a_i as possible through judicious applications of adjoint maps to V . Suppose first that $a_1 \neq 0$ in (33). One may re-scale a_1 such that $a_1 = 1$. We start with the combination:

$$V^{(1)} = V_1 + a_2 V_2 + a_3 V_3 + a_4 V_4 \quad (34)$$

If we act on $V^{(1)}$ by $Ad(\exp(a_2 V_2))$, we can make the coefficient of V_2 vanish:

$$V^{(2)} = V_1 + a_3 V_3 + a_4 V_4 \quad (35)$$

Next, we act on $V^{(2)}$ by $Ad(\exp(\frac{a_4}{a_3} V_4))$ to cancel the coefficient of V_4 , leading to the operator:

$$V^{(3)} = V_1 + a_3 V_3 \quad (36)$$

Using the adjoint representation (32), no further simplification is possible. Consequently, the 1-dimensional subalgebra spanned by V with $a_1 \neq 0$ is equivalent to the one spanned by $V_1 + \beta V_3$, $\beta \in R$.

The remaining 1-dimensional subalgebras are spanned by operators with $a_1 = 0$ which have the expressions:

$$V^{(4)} = a_2 V_2 + a_3 V_3 + a_4 V_4 \quad (37)$$

Let us assume that $a_2 \neq 0$ and scale to make $a_2 = 1$. Now we act on $V^{(4)}$ by $Ad(\exp(\frac{a_4}{a_3} V_4))$ so that it is equivalent with the operator:

$$V^{(5)} = V_2 + a_3 V_3 \quad (38)$$

No further simplification is possible. Consequently, the 1-dimensional subalgebra spanned by V with $a_2 \neq 0$ is equivalent to the one spanned by $V_2 + \alpha V_3$, $\alpha \in R$.

If we consider the case $a_1 = a_2 = a_3 = 0$, $a_3 \neq 0$, $a_3 = 1$, the following generator is obtained:

$$V^{(6)} = V_3 + a_4 V_4 \quad (39)$$

Acting on $V^{(6)}$ by $Ad(\exp(a_4 V_4))$, we obtain the operator V_3 which represents the next subalgebra of the optimal system.

Finally, let us consider the last case $a_1 = a_2 = a_3 = 0$, $a_4 \neq 0$, $a_4 = 1$ in (33). Results the last subalgebra V_4 .

In conclusion, the optimal system of 1-dimensional subalgebras has the form:

$$\{V_2 + \alpha V_3, V_1 + \beta V_3, V_3, V_4\} \quad (40)$$

2.1.3 Invariant solutions for 2D Ricci flow

Let us pass now to the problem of the *invariant quantities*. We shall analyze the invariants associated with the optimal system of symmetry operators (40).

- The operator $V_2 + \alpha V_3$ from (40) has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{\alpha y} = \frac{du}{-\alpha u} \quad (41)$$

By integrating these equations result 3 invariants with expressions:

$$I_1 = t, \quad I_2 = ye^{-\alpha x}, \quad I_3 = yu \quad (42)$$

By introducing the similarity variable $z \equiv I_2 = ye^{-\alpha x}$, designating the invariant $I_3 = h(t, z)$ as a function of the other ones, the following solution is obtained:

$$u(t, x, y) = \frac{h(t, z)}{y} \quad (43)$$

Setting the derivatives of (43) into the Ricci equation (25), results the similarity reduced equation for $h(t, z)$ with the form:

$$h_t h^2 - \alpha z^2 h h_{2z} - \alpha z^2 h_z^2 + \alpha z h h_z = 0 \quad (44)$$

The solution of the previous equation is:

$$h(t, z) = -\frac{1}{2} \left(r_3 t + \frac{r_2 r_3}{2r_1} \right) \left(-1 + \tanh^2 \left(\frac{\sqrt{\alpha r_3} (r_4 - \ln z)}{2\alpha} \right) \right) \quad (45)$$

with α, r_1, r_2, r_3 arbitrary constants and z the similarity variable.

Consequently, the invariant solution corresponding to operator $V_2 + \alpha V_3$ has the final form:

$$u(t, x, y) = -\frac{1}{2y} \left(r_3 t + \frac{r_2 r_3}{2r_1} \right) \left(-1 + \tanh^2 \left(\frac{\sqrt{\alpha r_3} (r_4 - \alpha x + \ln y)}{2\alpha} \right) \right) \quad (46)$$

- The operator $V_1 + \beta V_3$ from (40) has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{x} = \frac{dy}{\beta y} = \frac{du}{-(1+\beta)u} \quad (47)$$

In this second case, following the same procedure, we obtain also 3 independent invariants with expressions:

$$I_1 = t, \quad I_2 = yx^{-\beta}, \quad I_3 = y^{(1+\beta)/\beta} u \quad (48)$$

By introducing the similarity variable $z \equiv I_2 = yx^{-\beta}$, designating the invariant $I_3 = h(t, z)$ as a function of the other ones, the following solution is obtained:

$$u(t, x, y) = h(t, z) y^{-(1+\beta)/\beta} \quad (49)$$

Setting the derivatives of (49) into the Ricci equation (25), results the following $(1+1)$ reduced equation for $h(t, z)$:

$$h_t h^2 z^{(-1/\beta-2)} + \beta h h_{2z} + \beta z^{-1} h h_z - \beta h_z^2 = 0 \quad (50)$$

The solution of the previous equation is:

$$h(t, z) = -\frac{(p_1 t + p_2)}{2p_3^2 p_1 \beta} z^{1/\beta} \left(-1 + \tanh^2 \left(\frac{p_4 \beta - \ln(z)}{2p_3 \beta} \right) \right) \quad (51)$$

with $\beta, p_1, p_2, p_3, p_4$ arbitrary constants and z the similarity variable.

Consequently, by using (49) the invariant solution corresponding to the operator $V_1 + \beta V_3$ has the final form:

$$u(t, x, y) = -\frac{1}{2p_3^2 p_1 \beta} \frac{(p_1 t + p_2)}{xy} \left(-1 + \tanh^2 \left(\frac{p_4 \beta - \ln y + \beta \ln(x)}{2p_3 \beta} \right) \right) \quad (52)$$

- Because (25) is symmetric in x and y , there is also a second similarity solution of the form:

$$u(x, y) = \frac{g_3(x)}{y}, \quad \forall g_3(x) \quad (53)$$

which is generated by the symmetry operator V_3 from (30).

- Again, by the reason of symmetry in x and y of the analyzed model (25), the last similarity solution associated to the symmetry operator V_4 from (30), is generated as below:

$$u(x) = g_4(x), \quad \forall g_4(x) \quad (54)$$

2.1.4 The class of equations with Ricci type symmetries

Now, our aim is to find the class of equations with generic form (12) which admit Ricci type symmetries (28). Consequently, we have to solve the system (18) taking as unknown functions $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ and imposing a concrete form of the symmetry group.

Let us consider the *linear sector* for the Lie symmetries where coefficient functions of symmetry operators have the general expressions:

$$\varphi = 1, \quad \xi = mx + c_1, \quad \eta = vy + c_2, \quad \phi = ku + c_3 \quad (55)$$

with m, v, k, c_1, c_2, c_3 arbitrary constants.

Remark 1: For the concrete values $k = -(m + v)$, $c_3 = 0$ and nonvanishing values for all the other constants m, v, k, c_1, c_2 , the linearization (55) is reduced to (28) considered for the Ricci flow model. The results will be particularized for this case when the Ricci symmetries have been imposed.

With the choice (55), the general differential system (18) has for the "unknown" functions $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ solutions which can be expressed in terms of 14 arbitrary functions $\{\mathcal{F}'_j, \mathcal{F}''_j, j = 1, \dots, 7\}$. They are the general forms:

$$\begin{aligned} A(x, y, t, u) &= \mathcal{F}'_1(X', Y', Z') \xi^{\frac{m+v}{m}} + \mathcal{F}''_1(X'', Y'', Z'') \eta^{\frac{m+v}{v}} \\ B(x, y, t, u) &= \mathcal{F}'_2(X', Y', Z') \xi^{\frac{m+v-k}{m}} + \mathcal{F}''_2(X'', Y'', Z'') \eta^{\frac{m+v-k}{v}} \\ C(x, y, t, u) &= \mathcal{F}'_3(X', Y', Z') \xi^2 + \mathcal{F}''_3(X'', Y'', Z'') \eta^2 \\ D(x, y, t, u) &= \mathcal{F}'_4(X', Y', Z') \xi^{\frac{2v}{m}} + \mathcal{F}''_4(X'', Y'', Z'') \eta^{\frac{2m}{v}} \\ E(x, y, t, u) &= \mathcal{F}'_5(X', Y', Z') \xi^{\frac{v}{m}} + \mathcal{F}''_5(X'', Y'', Z'') \eta^{\frac{m}{v}} \\ F(x, y, t, u) &= \mathcal{F}'_6(X', Y', Z') \xi^{\frac{v}{m}} + \mathcal{F}''_6(X'', Y'', Z'') \eta^{\frac{m}{v}} \\ G(x, y, t, u) &= \mathcal{F}'_7(X', Y', Z') \xi^{\frac{k}{m}} + \mathcal{F}''_7(X'', Y'', Z'') \eta^{\frac{k}{v}} \end{aligned} \quad (56)$$

where

$$\begin{aligned} X' &\equiv \frac{\eta \xi^{(-\frac{v}{m})}}{v}, \quad Y' \equiv \frac{mt - \ln \xi}{m}, \quad Z' \equiv \frac{\phi \xi^{(-\frac{k}{m})}}{k}, \\ X'' &\equiv \frac{\xi \eta^{(-\frac{m}{v})}}{m}, \quad Y'' \equiv \frac{vt - \ln \eta}{v}, \quad Z'' \equiv \frac{\phi \eta^{(-\frac{k}{v})}}{k}. \end{aligned} \quad (57)$$

Remark 2: In the case $k = 0$, the arguments Z', Z'' are changing and they become:

$$Z' = \frac{mu - c_3 \ln \xi}{m}, \quad Z'' = \frac{vu - c_3 \ln \eta}{v}. \quad (58)$$

It is useful to choose particular solutions. Let us consider the cases:

Case1 : $c_3 = 0$ and $(m + v)$ is a multiple of the constant k , that is:

$$m + v = k \cdot n, \quad (\forall) n \in \mathbf{Z} \setminus \{0\} \quad (59)$$

Following the general solution (56)-(57), the concrete forms for $A(x, y, t, u)$, $B(x, y, t, u)$, $C(x, y, t, u)$, $D(x, y, t, u)$, $E(x, y, t, u)$, $F(x, y, t, u)$, $G(x, y, t, u)$ may be in this case:

$$\begin{aligned} A(x, y, t, u) &= \left[\frac{1}{2} u^n \xi^{(-\frac{n}{m}k)} \right] \xi^{(\frac{n}{m}k)} + \left[\frac{1}{2} u^n \eta^{(-\frac{n}{v}k)} \right] \eta^{(\frac{n}{v}k)} = u^n \\ B(x, y, t, u) &= \left[\frac{1}{2} n u^{(n-1)} \xi^{(-\frac{n-1}{m}k)} \right] \xi^{(\frac{n-1}{m}k)} + \left[\frac{1}{2} n u^{(n-1)} \eta^{(-\frac{n-1}{v}k)} \right] \eta^{(\frac{n-1}{v}k)} = n u^{(n-1)} \end{aligned} \quad (60)$$

$$C(x, y, t, u) = D(x, y, t, u) = E(x, y, t, u) = F(x, y, t, u) = G(x, y, t, u) = 0$$

where the parameter k has the form (59).

The 2D general equation (12) takes the particular form:

$$u_t = u^n u_{xy} + n u^{(n-1)} u_x u_y = (u^n u_x)_y \quad (61)$$

Remark 3: For $n = -1$, the previous equation generates the 2D Ricci flow model (25).

Case2 : $k = 0$, $c_3 = m + v$ and nonvanishing values for all the other constants m, v, c_1, c_2 .

The solution (56) could take in this case the form:

$$\begin{aligned} A(x, y, t, u) &= \left[\frac{1}{2} e^u \xi^{(-\frac{c}{m})} \right] \xi^{(\frac{c}{m})} + \left[\frac{1}{2} e^u \eta^{(-\frac{c}{v})} \right] \eta^{(\frac{c}{v})} = e^u \\ B(x, y, t, u) &= \left[\frac{1}{2} e^u \xi^{(-\frac{c}{m})} \right] \xi^{(\frac{c}{m})} + \left[\frac{1}{2} e^u \eta^{(-\frac{c}{v})} \right] \eta^{(\frac{c}{v})} = \frac{dA}{du} = e^u \\ C(x, y, t, u) &= D(x, y, t, u) = E(x, y, t, u) = F(x, y, t, u) = G(x, y, t, u) = 0 \end{aligned} \quad (62)$$

In this second case, the following 2D evolution equation of type (12) is generated:

$$u_t = e^u u_{xy} + e^u u_x u_y = (e^u u_x)_y \quad (63)$$

Remark 4: The equations (61) and (63) generated by the previous two cases correspond to the 2D nonlinear heat equation which has the general expression [20]:

$$u_t = (g(u) u_x)_y \quad (64)$$

It was proven [21] that the choices $g(u) = u^n$ and $g(u) = e^u$ are the only two possible cases for which the Lie symmetries exist and more the Lie operators have linear forms.

2.2 The Lie symmetry problems for 2D convective-diffusion equation

The second nonlinear application is represented by the 2D convective-diffusion equation [?]. It is a parabolic partial differential equation, which describes physical phenomena where particles or energy (or other physical quantities) are transferred inside a physical system due to two processes: diffusion and convection. In the simpler case when the diffusion coefficient is variable, the convection velocity is constant and there are no sources or sinks, the equation takes the form:

$$u_t = uu_{2x} + uu_{2y} - vu_x \quad (65)$$

with diffusion coefficient u and convective velocity $v = \text{const.}$ belongs to the Ox direction.

It is easy to remark that (65) results from the general class of equations (12) by choosing the particular functions:

$$\begin{aligned} C(x, y, t, u) &= D(x, y, t, u) = u, \quad F(x, y, t, u) = -v \\ A(x, y, t, u) &= B(x, y, t, u) = E(x, y, t, u) = G(x, y, t, u) \equiv 0 \end{aligned} \quad (66)$$

2.2.1 Lie symmetries for 2D convective-diffusion equation

In the conditions (66) the general determining system (18) for symmetries becomes:

$$\begin{aligned}
\phi_{2u} &= 0 \\
\xi_y + \eta_x &= 0 \\
2u\xi_x - \phi &= 0 \\
2u\eta_y - \phi &= 0 \\
-\eta_t - v\eta_x + u\eta_{2x} + u\eta_{2y} - 2u\phi_{yu} &= 0 \\
-\xi_t - v\xi_x + u\xi_{2x} + u\xi_{2y} - 2u\phi_{xu} &= 0 \\
\phi_t + v\phi_x - u\phi_{2x} - u\phi_{2y} &= 0
\end{aligned} \tag{67}$$

It has the solution:

$$\xi = \frac{c_1}{2}(x - vt) + c_2y + c_3, \quad \eta = \frac{c_1}{2}y - c_2(x - vt) + c_4, \quad \phi = c_1u \tag{68}$$

In this case, the Lie symmetry generator takes the expression:

$$U(x, y, t, u) = \frac{\partial}{\partial t} + \left(\frac{c_1}{2}(x - vt) + c_2y + c_3 \right) \frac{\partial}{\partial x} + \left(\frac{c_1}{2}y - c_2(x - vt) + c_4 \right) \frac{\partial}{\partial y} + c_1u \frac{\partial}{\partial u} \tag{69}$$

Consequently, the nonlinear convective-diffusion equation (65) admits the 4-dimensional Lie algebra spanned by the operators shown below:

$$\begin{aligned}
V_1 &= \left(\frac{x - vt}{2} \right) \frac{\partial}{\partial x} + \left(\frac{y}{2} \right) \frac{\partial}{\partial y} + u \frac{\partial}{\partial u}, \\
V_2 &= y \frac{\partial}{\partial x} - (x - vt) \frac{\partial}{\partial y}, \quad V_3 = \frac{\partial}{\partial x}, \quad V_4 = \frac{\partial}{\partial y}
\end{aligned} \tag{70}$$

When the Lie algebra of these operators is computed, the only non-vanishing relations are:

$$[V_3, V_1] = \frac{1}{2}V_3, \quad [V_4, V_1] = V_4, \quad [V_2, V_3] = V_4, \quad [V_4, V_2] = V_3 \tag{71}$$

2.2.2 Optimal system for convective-diffusion equation

For this model the commutators of the symmetry operators (70) are given below in the Table 2:

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	$-V_3/2$	V_4
V_2	0	0	V_4	$-V_3$
V_3	$V_3/2$	$-V_4$	0	0
V_4	V_4	V_3	0	0

Table 2: Lie brackets of the admitted symmetry

algebra

Let us consider the linear combination of the symmetry generators:

$$V = b_1V_1 + b_2V_2 + b_3V_3 + b_4V_4 \tag{72}$$

Our task is to simplify as many of the coefficients b_i as possible through judicious applications of adjoint maps to V . Suppose first that $b_1 \neq 0$ in (72). One may re-scale b_1 such that $b_1 = 1$. We start with the combination:

$$V^{(1)} = V_1 + b_2V_2 + b_3V_3 + b_4V_4 \tag{73}$$

If we act on $V^{(1)}$ by $Ad(\exp(2b_3V_3))$, we can make the coefficient of V_3 vanish and we obtain the operator:

$$V^{(2)} = V_1 + b_2V_2 + b'_4V_4, \quad b'_4 = b_4 + 2b_2b_3 \quad (74)$$

Using the adjoint representation (32) for our model, no further simplification is possible. Consequently, the 1-dimensional subalgebra spanned by V with $b_1 \neq 0$ is equivalent to the one spanned by $V_1 + \alpha V_2 + \beta V_4, \forall \alpha, \beta \in R$.

The remaining 1-dimensional subalgebras are spanned by operators with $b_1 = 0$ which have the expressions:

$$V^{(3)} = b_2V_2 + b_3V_3 + b_4V_4 \quad (75)$$

Let us assume that $b_2 \neq 0$ and scale to make $b_2 = 1$. Now we act on $V^{(3)}$ by $Ad(\exp(b_3V_4))$ so that it is equivalent with the operator:

$$V^{(4)} = V_2 + b_4V_4 \quad (76)$$

Here, further simplification is possible. If we act on $V^{(4)}$ by $Ad(\exp(-b_4V_3))$. In this case, we obtain the operator V_2 which is the following 1-dimensional subalgebra spanned V with $b_1 = 0$ and $b_2 = 1$.

If we consider the case $b_1 = b_2 = 0, b_3 \neq 0, b_3 = 1$, the following generator is obtained:

$$V^{(3)} = V_3 + b_4V_4 \quad (77)$$

If we act on $V^{(3)}$ by $Ad(\exp(\varepsilon V_2))$, where ε is the solution of the equation

$$\frac{b_4}{2}\varepsilon^2 + \varepsilon - b_4 = 0 \quad (78)$$

we can vanish the coefficient of V_4 and we obtain the operator:

$$V^{(4)} = b'_3V_3, \quad b'_3 = 1 + b_4\varepsilon - \frac{\varepsilon^2}{2} \quad (79)$$

where ε verified (78).

Consequently, by setting $b_1 = b_2 = 0, b_3 = 1$ in (72) is generated V_3 which is the last subalgebra of the optimal system.

In conclusion, the optimal system of 1-dimensional subalgebras for 2D convective-diffusion equation is:

$$\{V_2, V_3, V_1 + \alpha V_2 + \beta V_4, \forall \alpha, \beta \in R\} \quad (80)$$

2.2.3 Invariant solutions for the convective-diffusion equation

Through the reduced similarity method, each operator $\{V_i, i = \overline{1, 4}\}$ can generate invariant solutions of the model. Let us illustrate for our case what are the concrete forms of the similarity solutions generated not by this base of operators, but by the set of the optimal 1D subalgebras (80).

- Taking into account (70), the symmetry operator V_2 has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{vt - x} = \frac{du}{0} \quad (81)$$

In this second case, following the same procedure, we obtain also 3 independent invariants with expressions:

$$I_1 = t, \quad I_2 = vtx - \frac{x^2}{2} - \frac{y^2}{2}, \quad I_3 = u \quad (82)$$

With the notation $I_2 \equiv z$ and $I_3 = u \equiv g(t, z)$, the reduced equation for $g(t, z)$ will take the form:

$$g_t + (2z - v^2 t^2) g g_{2z} + 2g g_z + v^2 t g_z = 0 \quad (83)$$

It admits the solution:

$$g(t, z) = \frac{2z - v^2 t^2 + 2q_1}{4t + 2q_2} \quad (84)$$

where q_1, q_2, v are arbitrary constants.

Thereby, the second similarity solution corresponding to operator V_2 has the final form:

$$u(t, x, y) = \frac{2vtx - x^2 - y^2 - v^2 t^2 + 2q_1}{4t + 2q_2} \quad (85)$$

- The operator V_3 from (70) yields the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{0} = \frac{du}{0} \quad (86)$$

Therefore also 3 invariants are generated:

$$I_1 = t, \quad I_2 = y, \quad I_3 = u \quad (87)$$

Once again, expressing the last invariant I_3 as a function of the others ones, we obtain the third similarity solution:

$$u(t, y) = \frac{\frac{q_1}{2} y^2 + q_3 y + q_4}{q_2 - q_1 t} \quad (88)$$

with q_1, q_2 arbitrary constants.

- Again on the basis of (70), the last operator from (80), $V_1 + \alpha V_2 + \beta V_4$, has the characteristic equations:

$$\frac{dt}{0} = \frac{dx}{\alpha y + \frac{x-vt}{2}} = \frac{dy}{\frac{y}{2} + \alpha(vt - x) + \beta} = \frac{du}{u} \quad (89)$$

By integrating these equations one obtains 3 invariants with expressions:

$$I_1 = t, \quad I_2 = \frac{\frac{y}{2} + \alpha(vt - x) + \beta}{\frac{x}{2} + \alpha y - \frac{vt}{2}}, \quad I_3 = \frac{u}{\left[\frac{y^2}{2} + \alpha(vt - x) + \beta \right]^2} \quad (90)$$

By introducing the similarity variable $z \equiv I_2$, designating the invariant $I_3 = h(t, z)$ as a function of the other ones, the following solution is obtained:

$$u(t, x, y) = h(t, z) \left[\frac{y^2}{2} + \alpha(vt - x) + \beta \right]^2 \quad (91)$$

Setting the derivatives of (91) into the convective-diffusion equation (65), we obtain the following $(1+1)$ reduced equation for $h(t, z)$:

$$h_t - 2 \left(\alpha^2 + \frac{1}{4} \right) z^3 h h_z - 4 \left(\alpha^2 + \frac{1}{4} \right) z h h_z - 2 \left(\alpha^2 + \frac{1}{4} \right) h^2 = 0 \quad (92)$$

The solution of the previous equation is:

$$h(t, z) = \frac{-1}{2 \left(\alpha^2 + \frac{1}{4} \right) t - \gamma} \quad (93)$$

with α, γ arbitrary constants.

Using (91), the invariant solution generated by the operator $V_1 + \alpha V_2 + \beta V_4$ is pointed out:

$$u(t, x, y) = -\frac{1}{2\left(\alpha^2 + \frac{1}{4}\right)t - \gamma} \left[\frac{y^2}{2} + \alpha(vt - x) + \beta \right]^2 \quad (94)$$

where α, β, γ, v arbitrary constants.

2.2.4 Inverse symmetry problem for 2D convective-diffusion equation

Our aim is now to find the class of equations with generic form (12) which admits the same symmetries with those corresponding to 2D nonlinear convective-diffusion equation (65). Consequently, we have to impose that the coefficient functions (68) which determine the base of symmetry operators (70) verify the general determining system (18).

The solutions of differential system (18) describe the coefficient functions of the general evolutionary equation (12) as follows:

$$\begin{aligned} A &= B = 0, \quad C = D = c_3 u, \\ E(u) &= \sqrt{u} \left[c_4 \cos \left(\frac{c_2}{c_1} \ln(u) \right) - c_5 \sin \left(\frac{c_2}{c_1} \ln(u) \right) \right] \\ F(u) &= \sqrt{u} \left[c_4 \sin \left(\frac{c_2}{c_1} \ln(u) \right) + c_5 \cos \left(\frac{c_2}{c_1} \ln(u) \right) - v \right] \\ G(u) &= c_6 u \end{aligned} \quad (95)$$

where c_j , $j = \overline{1, 6}$ and v are arbitrary constants.

In particular, for $c_3 = 1$, $c_4 = c_5 = c_6 = 0$ and arbitrary c_1 and c_2 , the solution (95) generates the 2D nonlinear convective-diffusion equation (65) discussed above.

3 Conclusions

This lecture intended to present some key aspects on how a dynamical systems whose evolution is described by a nonlinear differential equation can be studied using the symmetry method. The main steps which have to be done in order to find a set of exact solutions are: (i) determination of the general form for the symmetry operator; (ii) determination of the optimal set of independent operators which can generate the minimal subalgebras; (iii) based on the optimal set of independent operators and using the similarity reduction procedure, a complete set of invariant solutions can be generated; (iv) last but not least, a special method can be applied in order to find the largest class of nonlinear differential equations which belong to the same class as a given equation in the sense of the symmetries they observe. This algorithm was applied for two important examples of nonlinear 2D partial derivative equations, the Ricci flow and the convective-diffusion equation. For the first example the optimal system of subalgebras contains the same number of generators, four, as the whole symmetry algebra. The optimal system of symmetry subalgebra for the convective-diffusion equation has the dimension three, despite the existence of four independent symmetry operators. In both cases, the whole set of invariant solutions had been obtained.

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